ECE 4331:
Linear Systems & Signals Laboratory

Experiment 5

Circuit Realization of the Lorenz Chaotic System

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I have neither received nor provided any assistance on this work.

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I: Introduction:

The purpose of ECE 4331 Experiment 5 was to study and get acquainted with the chaotic behavior and dynamics of the diffusion-less Lorenz system. The objectives of this experiment were to realize the diffusion-less Lorenz system using analog circuit components and analyze the experimental behavior of the system using the available laboratory equipment and the understanding of such systems provided to us in ECE 4330 lectures.

In this experiment, our exploration into linear systems and electronic circuit analysis advanced into the realm of dynamic, non-linear systems. The diffusion-less Lorenz system, a simplified variant of the classical Lorenz system, is known for its chaotic behavior, making it a fascinating subject for study in electrical and computer engineering. By employing a range of electronic components and leveraging the principles of linear systems we have studied thus far, we aimed to recreate the system's characteristic dynamics experimentally.

The implementation involved careful consideration of component selection and circuit topology to ensure that the circuit accurately represented the theoretical model provided to us. This hands-on approach provided valuable insights into the practical challenges of circuit design and realization when designing a larger model than we had accomplished in previous experiments.

This report details the procedures followed in the experiment, the setup and configuration of the circuit, and the analysis of the results obtained. We will discuss how the circuit's behavior correlated with the theoretical predictions of the diffusion-less Lorenz system and examine any discrepancies between theoretical expectations and experimental observations. All experimental values and results obtained will be compared to their theoretical values by calculation of their relative error, whenever possible.

II: Procedure:

II.a: Components Used:

In the completion of experiment 5, the following components and laboratory tools were used:

- *RIGOL DG1022* Waveform Generator
- RIGOL MSO1104Z Oscilloscope
- RIGOL DM3058 digital multimeter
- OPA277 precision operational amplifier (5)
- AD633 analog multiplier (2)
- 10kΩ (1/4 Watt, 1%) resistor (2)
- 100 kΩ (1/4 Watt, 1%) resistor (1)
- 1MΩ (1/4 Watt, 1%) resistor (3)
- 10MΩ (1/4 Watt, 1%) resistor (3)
- 5.1kΩ (1/4 Watt, 5%) resistor (1)
- 10kΩ (1/4 Watt, 5%) resistor (1)
- 50kΩ, 10-turn precision potentiometer

- 100pF (Polypropylene film, 1%) capacitor (3)
- 0.01μF (Polypropylene film, 5%) capacitor (3)
- 0.1μF (Polypropylene film, 5%) capacitor (3)

II.b: Pre-Laboratory Procedure:

Figure 1: Diffusionless Lorenz Model

$$\frac{dx}{dt} = y - x \qquad\qquad x = \int (y - x)dt$$

$$\frac{dy}{dt} = -xz \quad \textbf{or in integral form} \rightarrow \quad y = \int -xz\,dt$$

$$\frac{dz}{dt} = xy - a \qquad\qquad z = \int (xy - a)dt$$

The following steps described in this subsection were completed in the fulfillment of the pre-laboratory activity for experiment 5:

a) Using Simulink, create a block diagram that realizes the diffusion-less Lorentz system and verify its behavior.

b) Using the created block, Set a = 1, integrators initial conditions to (x(0), y(0), z(0)) = (1,1,1) and set the simulation time to 400 sec. Generate a plot of $x(t)$ and $z(t)$ versus time. Also, generate phase-plane trajectories for $z(t)$ vs $x(t)$, $y(t)$ vs $x(t)$ and $z(t)$ vs $y(t)$. NOTE: Make sure the Model Settings in Simulink are set correctly and as shown in the prelab.

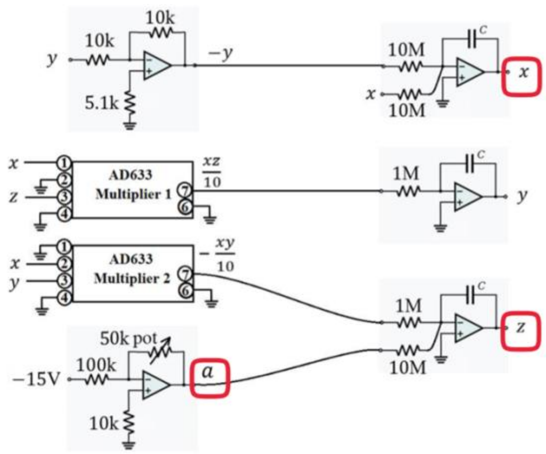
c) Repeat the simulations with a = 1.88 and a = 2.5 after setting integrator initial conditions to (1,0,0) and setting the simulation time to 500. Generate $z(t)$ vs $x(t)$ plots for each case.

d) Employ Simulink to demonstrate the chaotic behavior of the system for a = 1. You can do that by simulating two systems that are identical except that one starts from the initial state (1,1,1) and the other starts from (1.001,1,1). Generate a plot of $x1(t)$ and $x2(t)$, set simulation time to 50 seconds.

II.c: Laboratory Activity Procedure:

The following steps described in this subsection were completed during the laboratory activity session for experiment 5:

a) Use the breadboard to build the following circuit (shown below). For the first few steps of the lab activity, use three 100pF, 1% tolerance Polypropylene capacitors.



Make sure that all seven chips have the proper ±15V supply connections

b) Connect the output signal $x(t)$ to CH1 of the scope. Connect the output signal $z(t)$ to CH2 of the scope. Set both channels to 1.5V/div. Set the time scale to 5ms/div. Set the scope to the X-Y (full display) plotting mode. Power up the circuit. The noise in the circuit will be exploited to trigger the dynamics of the system being simulated (this provides the initial conditions!).

c) Use a voltmeter to monitor the voltage that sets the parameter a. This is the output of the inverting amplifier with the precision potentiometer. Adjust the potentiometer so that the voltmeter reads 1 Volt (i.e., a=1). Save three captures of the scope's traces.

d) Switch the circuit power off. Replace the integrator capacitors to the 0.01μF (5%) Polypropylene film capacitors. Power up your circuit. What effect does using the larger capacitors have on the solution? Provide an explanation for these effects. Save three captures of the scope's traces.

e) Repeat the previous step using 0.1μF (5%) Polypropylene film capacitors.

f) Reinstall the 100pF integrator capacitors. Adjust the potentiometer so that you set a close to 1.85. Fine tune the potentiometer until a stable periodic orbit (Similar to the one generated in the pre-lab simulations) is generated. Record the value of a. Save the scope's trace image of the output signal.

g) Switch the power off, then switch it back on. Does the circuit converge to the same limit-cycle? Does the trajectory agree with the Pre-lab Simulink simulations? Explain.

h) Repeat Steps f and g for a = 2.5. Fine tune the potentiometer until a stable periodic orbit is generated once again. Save the scope's trace image.

i) With a tuned as in Step H, generate and save the phase plane traces for $y(t)$ vs x(t). Repeat for z(t) vs y(t). Additionally, Obtain and save the plots for $z(t)$ vs $x(t)$ and $y(t)$ vs $x(t)$ with $a \cong 5.2$. Finally, try to obtain the plot $z(t)$ vs $x(t)$ for the oscillations (that are difficult to get) in the vicinity of $a \cong 0.4$ and $a \cong 0.55$. By studying the bifurcation diagram provided, it can be observed that these values of a should yield phase-plane trajectories that are periodically stable.

III: Results:

The following sections describe and analyze in detail the theoretical and experimental results obtained during the completion of the pre-laboratory and the laboratory activity. Many of the circuit elements' theoretical vs. experimental values, and their relative error, are tabulated in these sections. Also, certain key findings and values may be highlighted and boxed.

III.a: Pre-Laboratory Results:

The following section provides the results of the pre-laboratory tasks. By utilizing MATLAB's Simulink software, simulations of the behavior output signals of the Lorenz diffusion-less system were successfully completed. The results and preliminary analysis of the system are provided below.

<u>Step A: Simulink Block Diagram</u>

Figure 2: Simulink Block Diagram of Lorenz System

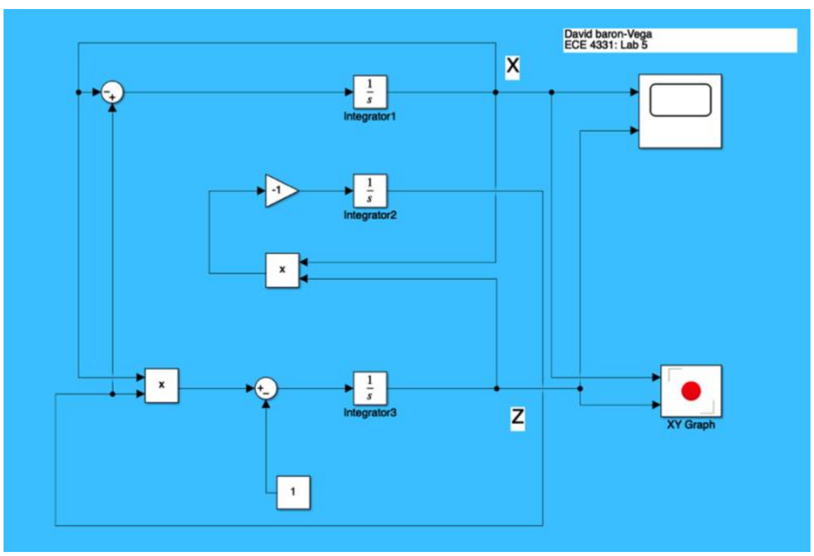


Figure 2 (above) shows the block-diagram realization of the diffusion-less Lorenz mathematical system. In the case of figure 2, the XY graph outputs the phase-plane trajectory of x(t) vs z(t). This same model, with slight adjustments to the output nodes, was used to depict the phase-plane trajectories of x(t) vs. y(t) and y(t) vs. z(t) as well.

Step B: Block Diagram Output

In step B, the simulation model created in step A was verified by generating the three characteristic phase-plane trajectories of the Lorenz system, as well as the graph that shows x(t) vs. z(t), both with respect to time. In all four of the graphical figures provided below, the chaotic, unpredictable behavior of the model is self-evident and shown in the butterfly-like shapes that are characteristic of this famous 3-dimensional Lorenz system. Figure 3 shows the model settings used in Simulink to generate all the proceeding simulations, for reference. Additionally, it is worth noting that MATLAB version R2021b was used in the development of these simulations.

Figure 3: Model Settings Used

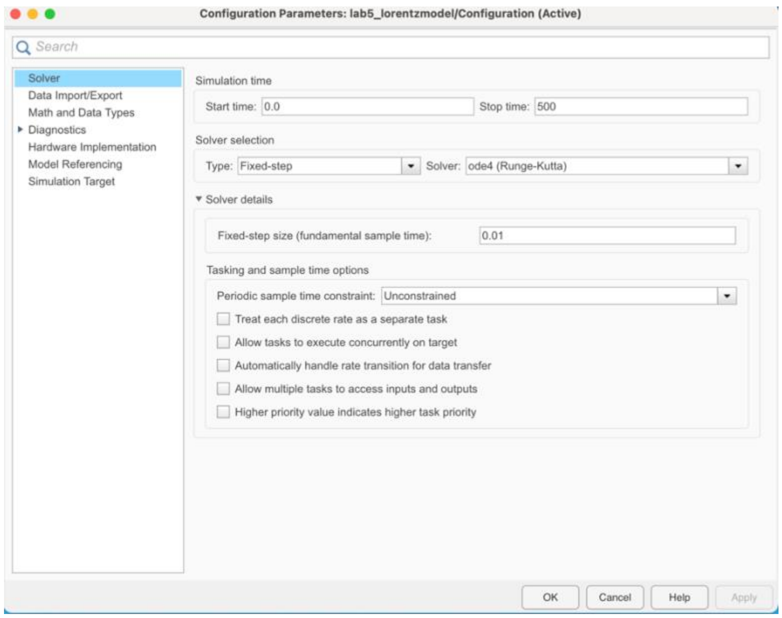


Figure 4: Signal Graph of x(t) (yellow) & z(t) (blue) vs. Time

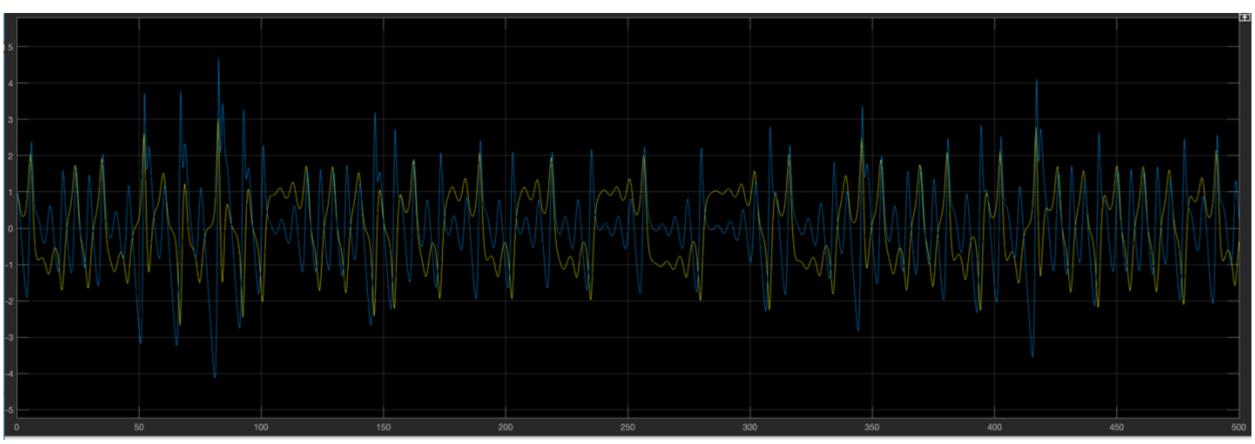


Figure 4 above shows the time-dependent signals of x(t) and z(t) of the Lorenz System, simulated for at least 400 seconds. The initial conditions of (x,y,z) = (1,1,1) were set by selecting each of the three integrators corresponding to each of the 3 outputs/variables and assigning their initial condition to 1. Additionally, the constant ‘a’ was set to 1 in this instance. The non-periodic, chaotic behavior of the system at this value of ‘a’ and at these initial conditions in evident by the above graph. Neither signal ever matches the other, and each signal never repeats.

<u>Phase-Plane Trajectories, A = 1, I.C.’s (x,y,z) = (1,1,1):</u>

Figure 5: x(t) vs. z(t)

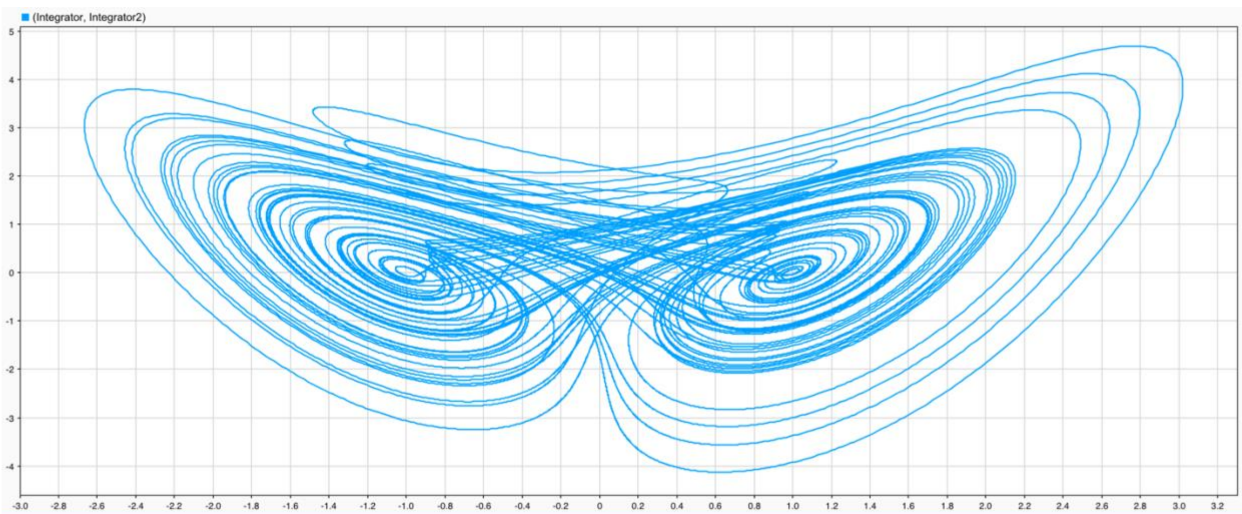


Figure 6: y(t) vs. z(t)

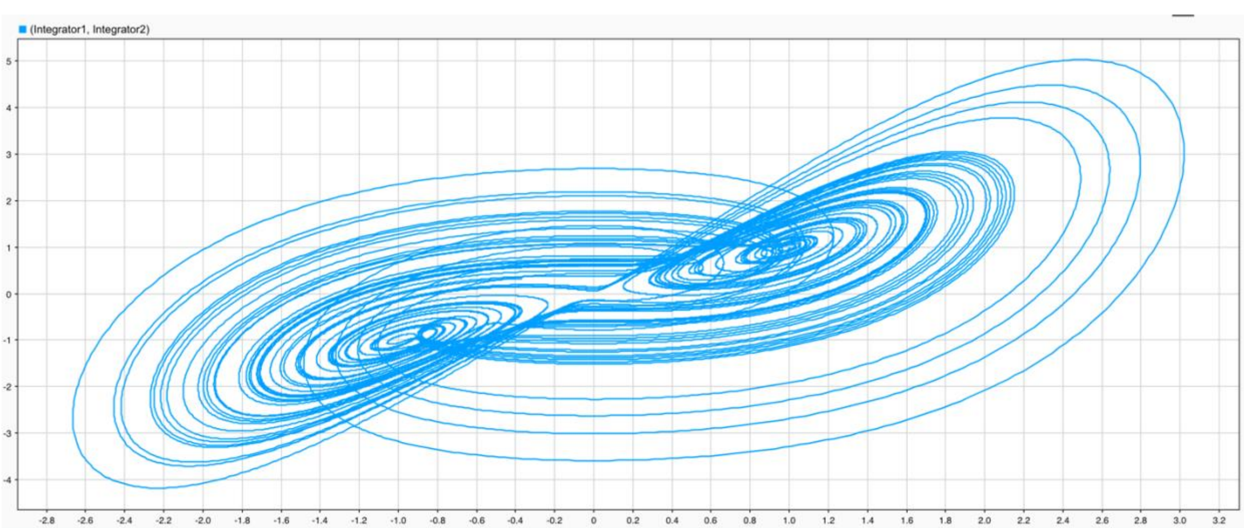
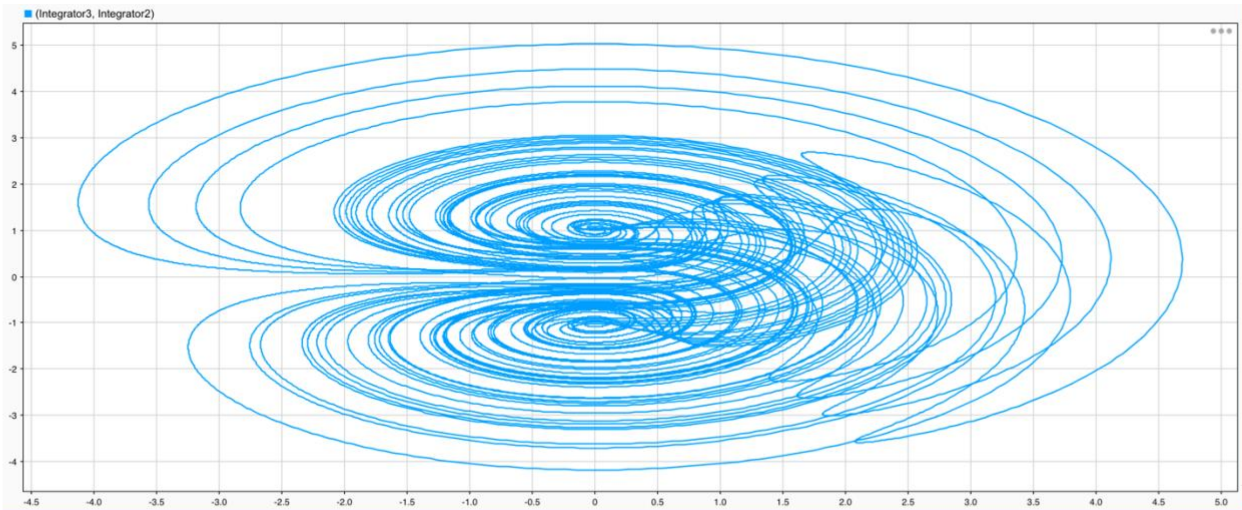


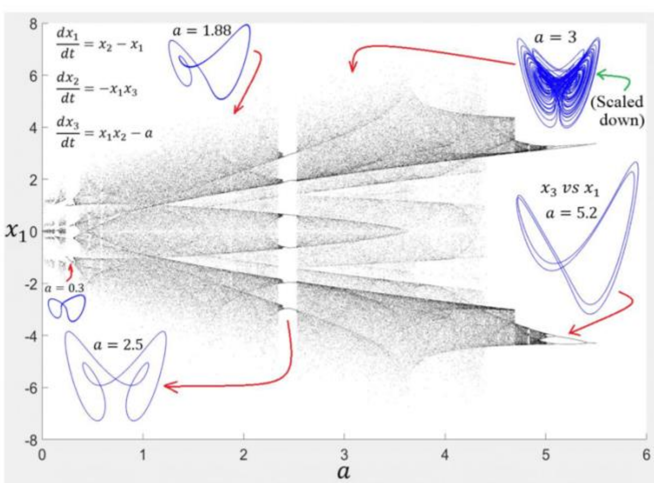
Figure 7: z(t) vs. y(t)



Step C: Observing Convergence for Certain Values of A:

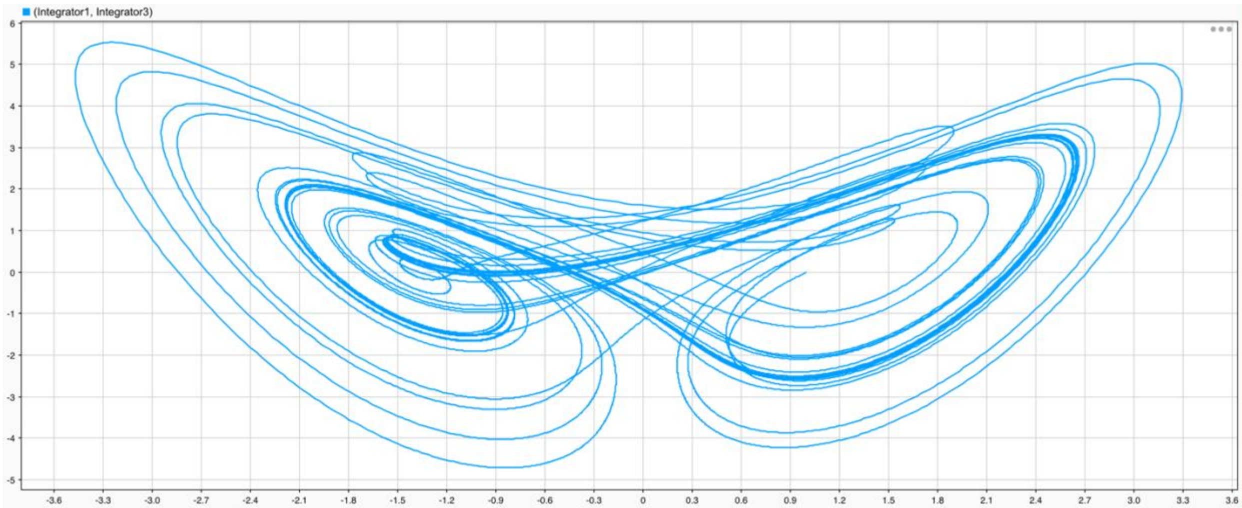
When certain values of constant ‘a’ are applied to the system, at specific initial conditions, the output of the phase-plane trajectories may reach periodic stability (very interesting!). The breakdown of values for x’s initial conditions for varying values of a at which the Lorenz system may exhibit stability is thoroughly summarized in the bifurcation diagram provided below:

Bifurcation Diagram of the (Diffusion-less) Lorenz Dynamical System



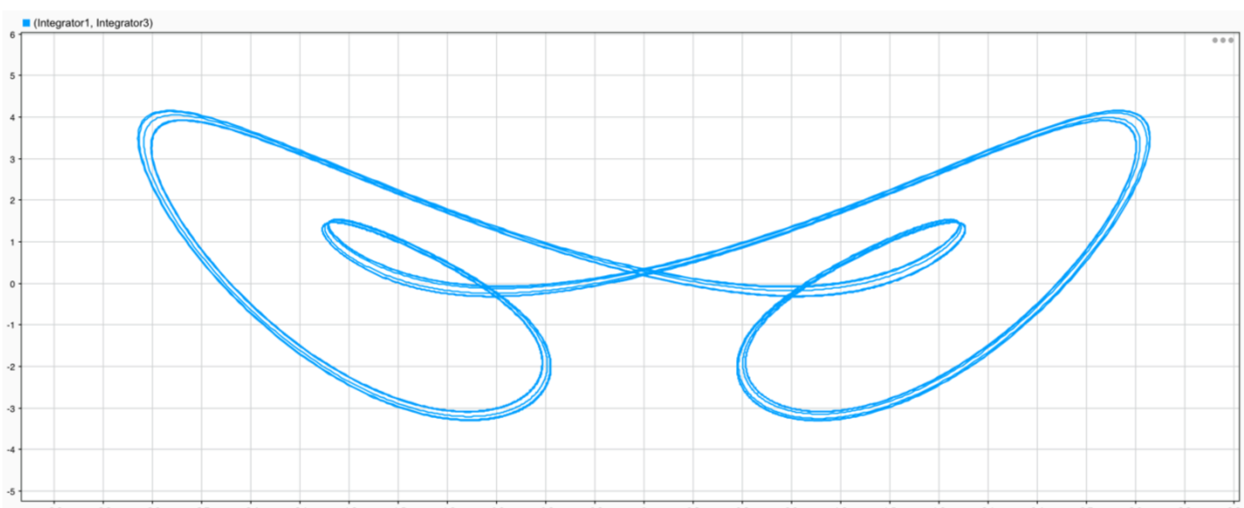
The points at which there are densely shaded spaces in the diagram below are points at which this periodic stability may be observed. Again, it is worth emphasizing that these results are very sensitive to the initial conditions within the system. The two figures below show a simulation of the Lorenz system reaching periodic stability at certain values for constant a, with initial conditions of (1,0,0).

Figure 8: A = 1.88, z(t) vs. x(t)



As shown in the figure above, when the constant A was set to 1.88, the system approaches some stability. Although there is still noticeable chaotic behavior, the density of curvature in a particular region matches the shape shown in the bifurcation diagram.

Figure 9: A = 2.5, z(t) vs. x(t)



In Figure 9, the phase-plane trajectory of the system with the constant A value set to 2.5 is shown. There is very clear convergence of the system at this value, which matches very clearly the results of the provided bifurcation diagram.

These simulated values for varying values of A which yield convergence within the system will be used to compare the experimental results obtained during the laboratory activity that also yielded convergence and stability.

Step D: Observing Large Difference in Output Signals with Slight Change in Initial Conditions:

A phenomenon characteristic to chaotic systems such as the Lorenz model is that a very small change in the system's initial conditions will yield vastly different outcomes the more time passes. This is essentially the definition of a chaotic system: unlike linear differential systems with analytical or numerical solutions, whose answers can be used to predict future outcomes when initial conditions are known, it is impossible to predict the long-term behavior of a chaotic system, even when initial conditions are known.

This 'butterfly effect' was demonstrated using a pair of identical block diagrams, same as the diagrams used in steps a, b, and c. One system was given initial conditions of (1,1,1), while the second system was provided with initial conditions of (1.001,1,1). The output signal x(t) of both systems were plotted together in respect to time. What we observe in Figure 11 is that, initially, the output signals of both models are similar, but as more time passes, the two signals take very different, unpredictable paths.

Figure 10: Pair of Lorenz Systems with Different Initial Conditions

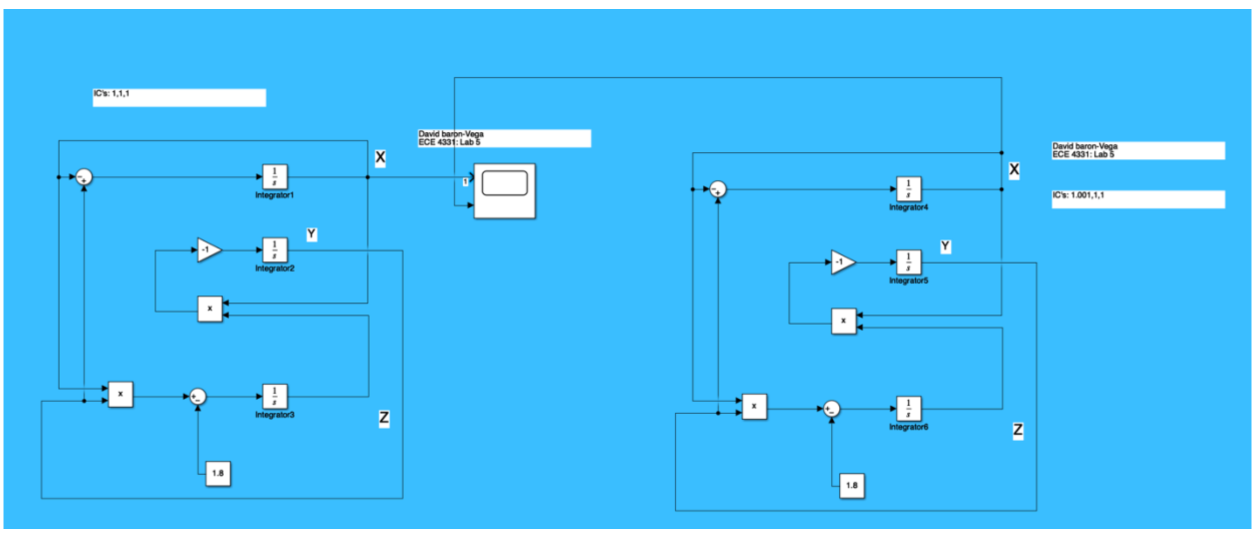
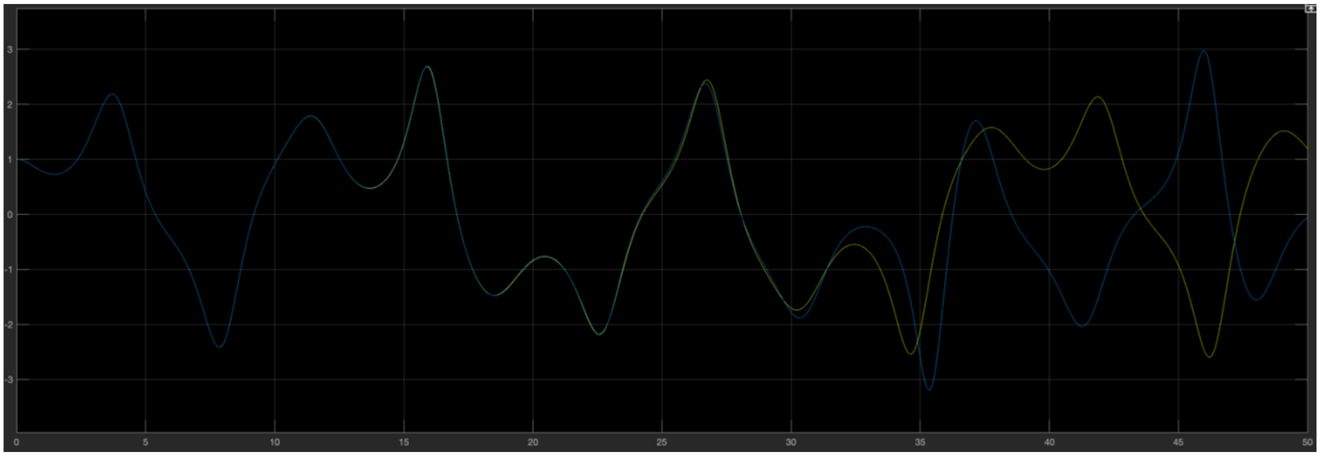


Figure 11: x1(t) & x2(t) vs. time



As can be seen in figure 11, it took only approximately 30 seconds for the two identical systems, with nearly identical initial conditions, to diverge from each other.
Fascinating!

III:b Laboratory Activity Results:

This section provides the results obtained during the completion of the laboratory activity, in person.
Table 1 and 2 below provide the experimental vs. theoretical values, and the relative error, of all the circuit components used in the completion of each of the proceeding steps. Table 1 tabulates this for all the different capacitors used, and Table 2 tabulates this for all resistive elements used.

Table 1: Percent Error of All Capacitors Used

Component Name	Theoretical Value	Experimental Value	Percent Error
C 1.a	100pF	95pF	-5.0%
C 2.a	100pF	96pF	-4.0%
C 3.a	100pF	96pF	-4.0%
C 1.d	0.01μF	9,797pF	-2.210%
C 2.d	0.01μF	9,605pF	-3.950%
C 3.d	0.01μF	9,772pF	-2.280%
C 1.d	0.1μF	97,777pF	-2.233%
C 2.d	0.1μF	98,437pF	-1.563%
C 3.d	0.1μF	98,045pF	-1.955%

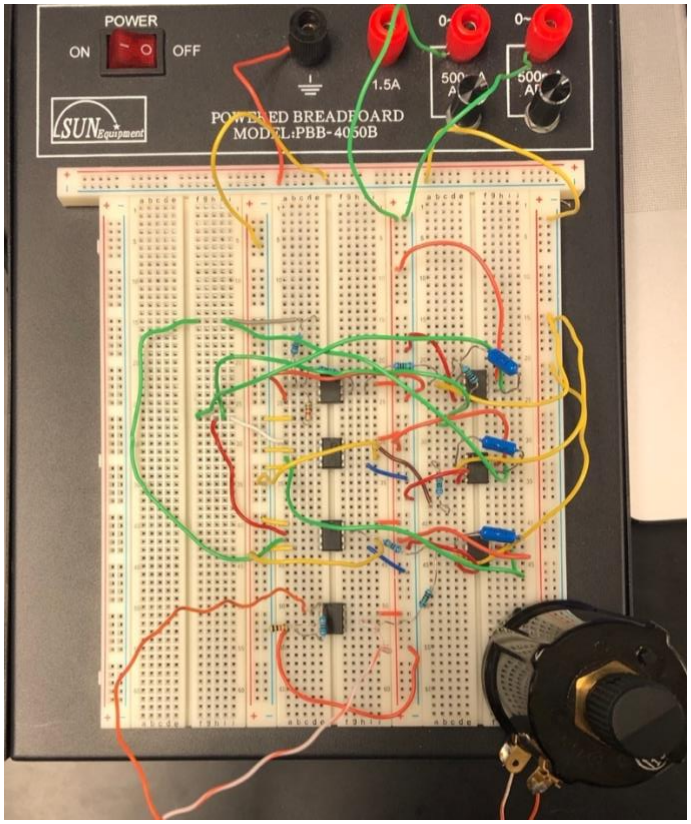
Table 2: Percent Error of All Resistors Used

Component Name	Theoretical Value	Experimental Value	Percent Error
R1.1	10MΩ	10.163MΩ	1.163%
R1.2	10MΩ	10.075MΩ	0.75%
R1.3	10MΩ	10.002MΩ	0.020%
R2.1	1MΩ	1.0067MΩ	0.670%
R2.2	1MΩ	1.076MΩ	7.60%
R2.3	1MΩ	0.995MΩ	-0.50%
R3	100KΩ	100.21KΩ	0.210%
R4.1	10KΩ	9.984KΩ	-0.160%
R4.2	10KΩ	9.936KΩ	-0.640%
R4.3	10KΩ	9.905KΩ	-0.950%
R5	5.1KΩ	5.077KΩ	-0.451%

Step A: Realizing Physical Circuit, Comparing Theoretical vs. Experimental Values of Components

The physical circuit designed during the laboratory session, which models the diffusion-less Lorenz system, is shown in Figure 12 below. Figure 12 shows the circuit with the 100pF capacitors at the integrators being used. The only changes made to the circuit shown throughout the completion of all the following steps was the swapping of the integrator's capacitors. These images were not included for redundancy, but it will be indicated for which steps these changes were made.

Figure 12: Physical Realization of Lorenz System



<u>Steps B and C: x(t) vs. z(t) for A = 1.0V</u>

Figure 13: Constant A set to 1.0V

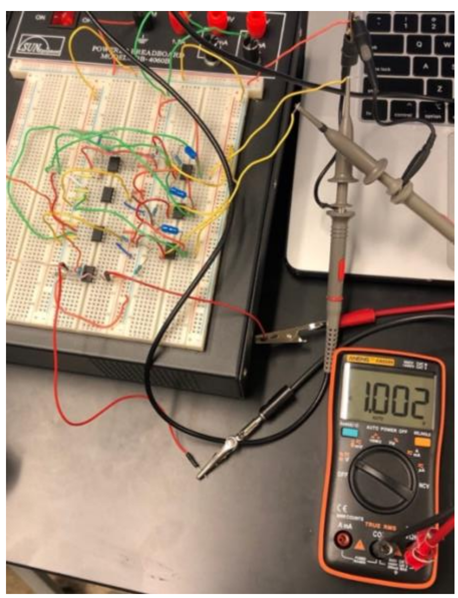
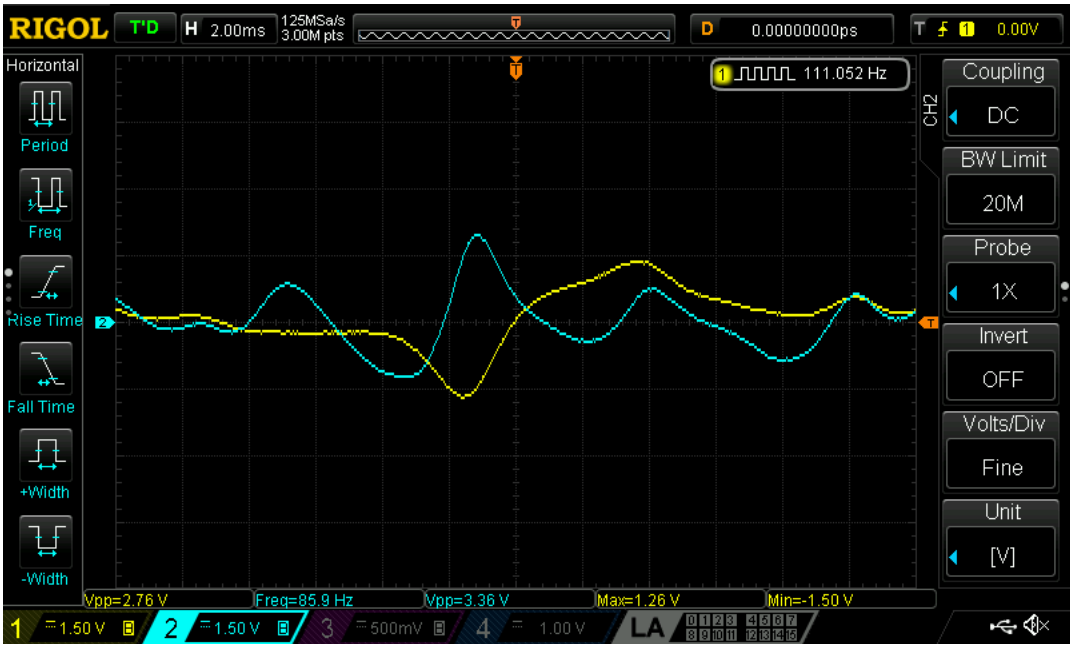


Figure 13 shows the realized Lorenz circuit with a multimeter measuring the voltage output of the inverting amplifier used in the system, which is the constant A in the system. The precision potentiometer was used as the feedback resistor in the inverting amplifier; by modulating the resistance in the potentiometer, the amplification factor across the inverting amplifier could be increased or decreased, allowing us to change and set the value of constant A with accuracy. Figure 13 shows the value of constant A measured to be approximately 1.0V.

Figure 14: x(t), z(t) vs. time



In Figure 15 above, we see the outputs x(t) (in yellow) and z(t) (in blue) of the Lorenz circuit, with constant A set to 1.0V. These experimental results demonstrate the aperiodic outputs of the system, which match the behavior of the simulated results.

The following three figures below show 3 separate screen-captures of the oscilloscope in X-Y mode, plotting x(t) vs. z(t) of the Lorenz circuit. These figures represent the phase-plane trajectories of the system, which are like the simulated results obtained. As can be seen, the system displays chaotic behavior for constant A = 1.0V. Although we do not manually select initial conditions for the circuit realization of the system, the internal noise of the electrical components provide the initial voltage necessary for kick the system into oscillatory behavior.

Figure 15: Phase Plane-Trajectory of x(t) vs. z(t) pt.1

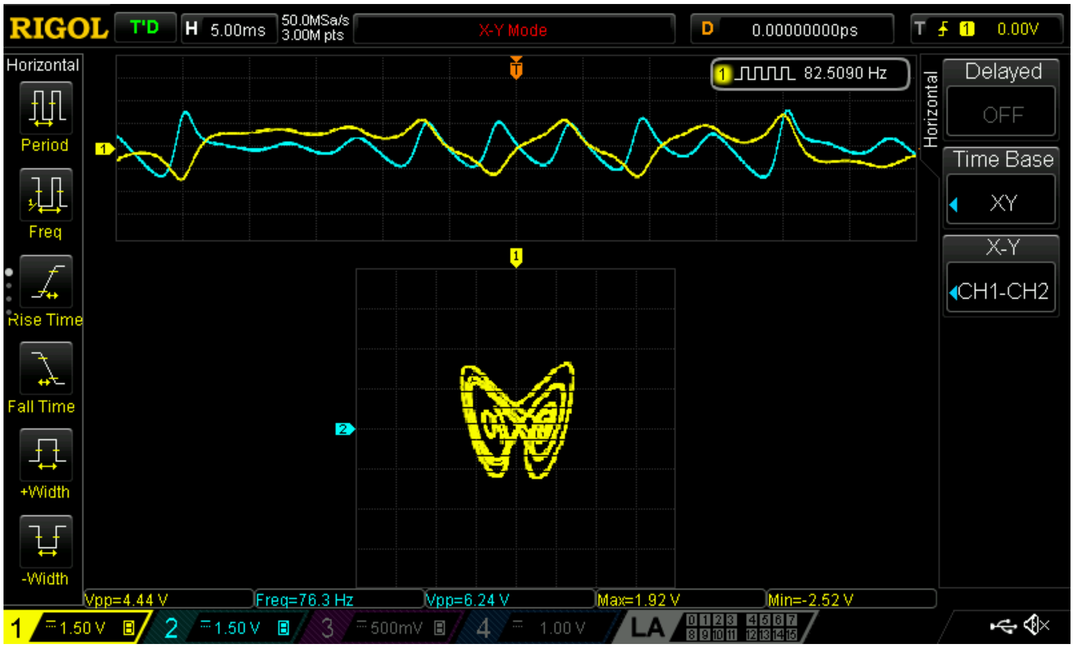


Figure 16: Phase Plane-Trajectory of x(t) vs. z(t) pt.2

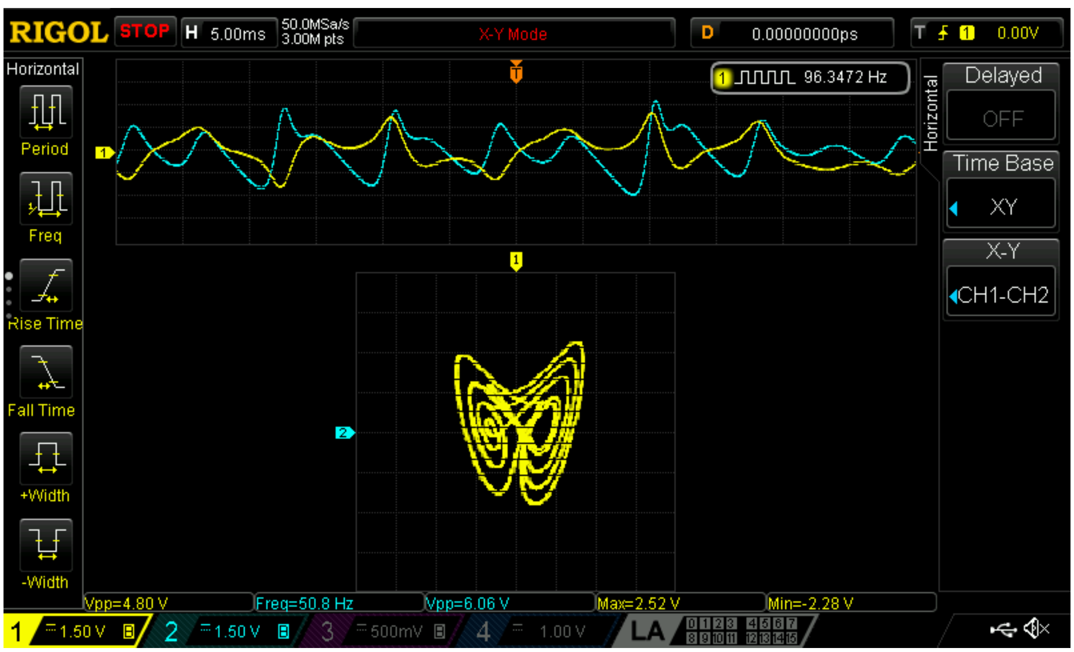
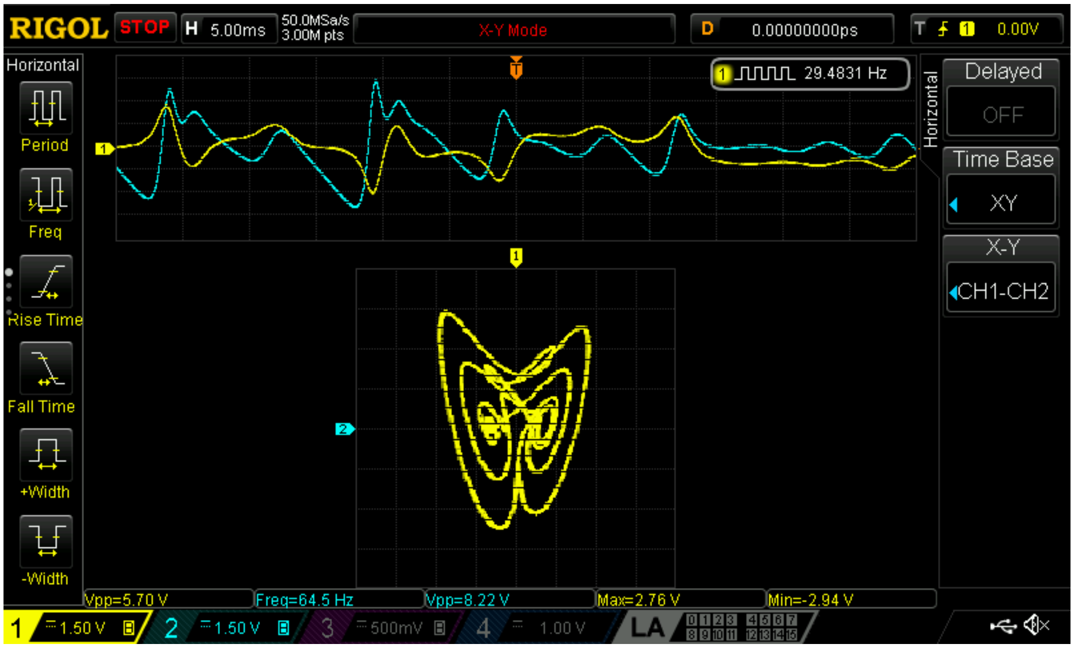


Figure 17: Phase Plane-Trajectory of x(t) vs. z(t) pt.3



<u>Step D: x(t) vs. z(t), A = 1.0V, Using 0.01µF Capacitors:</u>

Figures 18 and 19 below show the phase-plane trajectories of x(t) vs. y(t) of the Lorenz circuit, after having swapped the 100pF capacitors with 0.01µF Capacitors in the three integrating op-amps. As we have learned, the time constant of an integrator circuit is given by τ=RC, where R is the resistance and C is the capacitance of the op-amp. By increasing the capacitance, we increased the time constant value. This means the integrators, and therefore the entire circuit and its outputs responded slower to changes in the input signal.
Although the two figures below seem to indicate stability or convergence in the system, this was not the case. Because the system's resonant frequency was much lower than when it had the 100pF capacitors, the oscilloscope's trace was much slower and was less able to demonstrate the chaotic, aperiodic behavior of the output signals. Additionally, by increasing the capacitance across the integrators, the amplitude of the output signals, and therefore the phase-plane trajectory, was decreased. This can be understood by acknowledging the transfer function of an integrator circuit: *H*(*s*)= *Vout/Vin* = -1/RC. By increasing 'C', we decrease the amplitude of the output signal. As can be seen by the time-dependent graphs, at the top of figures 18 and 19, we can observe the decrease in amplitude of the signals x(t) and z(t).

Figure 18: Phase Plane-Trajectory of x(t) vs. z(t), 0.01µF Capacitors pt.1

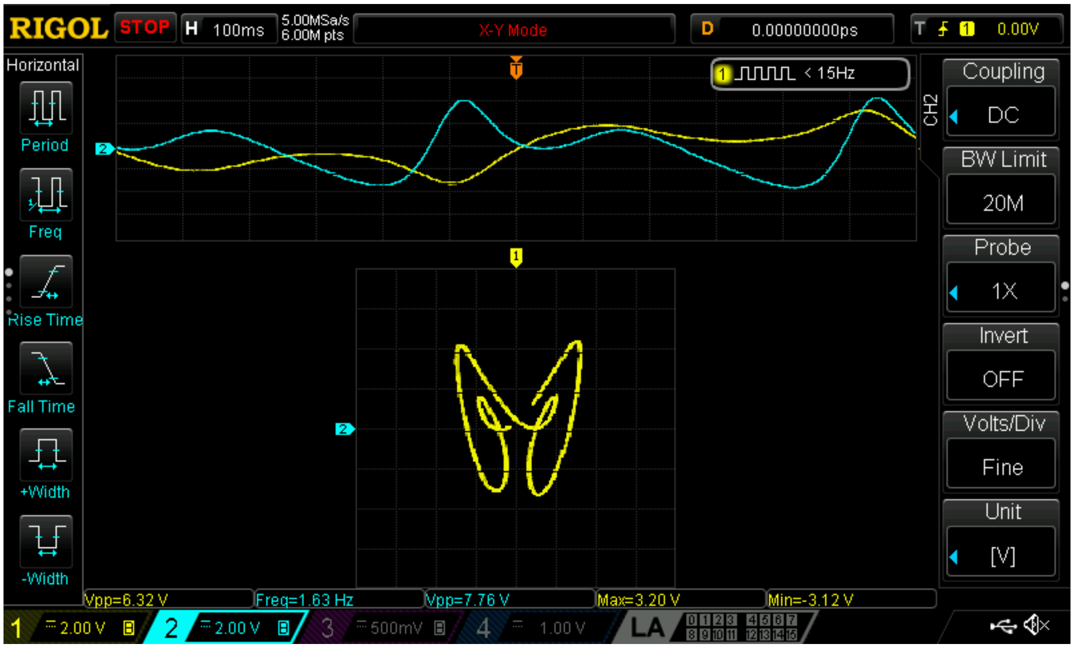
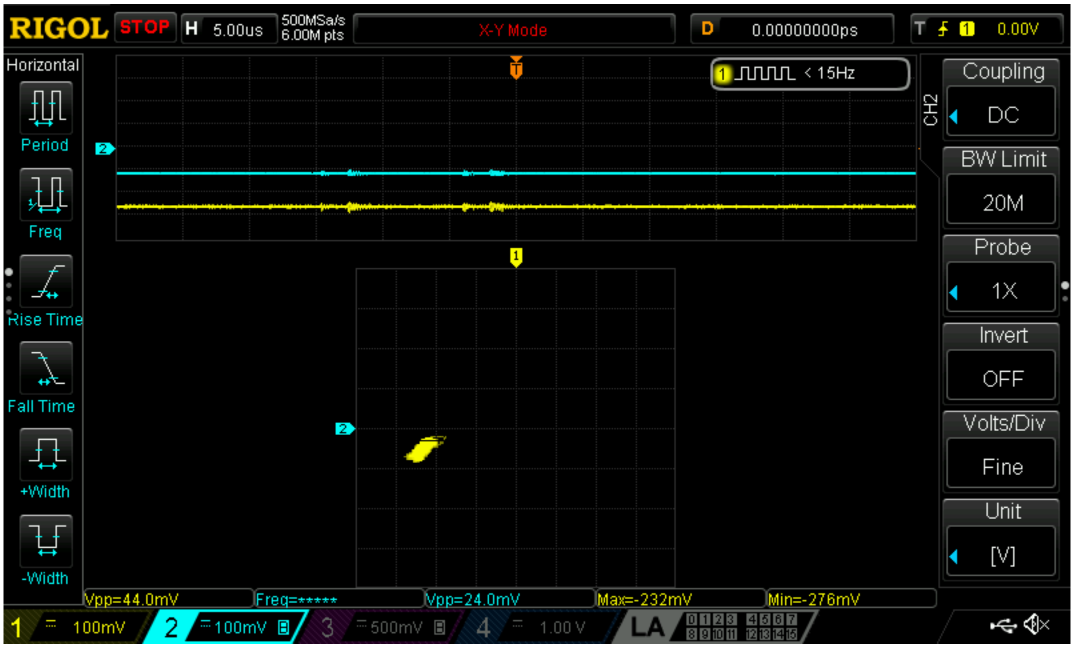


Figure 19: Phase Plane-Trajectory of x(t) vs. z(t), 0.01µF Capacitors pt.2

<u>Step E: x(t) vs. z(t), A = 1.0V, Using 0.1µF Capacitors:</u>

In step E, we once again increased the capacitance value of the integrating circuits in the Lorenz circuit. Figure 20 shows the resulting output of x(t) vs. z(t) and its phase-plane trajectory with these changes to the circuit. Once again, by increasing the capacitance in the integrating circuits, we decrease the resonant frequencies of the signal, and decrease the output amplitudes of x(t) and z(t). Unfortunately, after many attempts to tune the oscilloscope to these changes, we were unsuccessful in acquiring any clear or tangible output signal. The phase-plane needle was seen moving across the graph, but it did not move fast enough to witness any graph. The amplitudes of x(t) and z(t) seem to also have been completed minimized, into meaningless noise.

Figure 20: Phase Plane-Trajectory of x(t) vs. z(t), 0.1μF Capacitors



<u>Step F: x(t) vs. z(t), A = 1.85V, Using 100pF Capacitors:</u>

In step F, we once again use the 100pF capacitors across the integrating op-amps. By tuning the potentiometer, we changed the constant value of A until we witnessed the phase-plane trajectory of x(t) vs. z(t) approach stability. As shown in figure 21, the value of A that provided stability to the circuit was 1.852V. The resulting convergence in the phase-plane trajectory is shown in figure 22.

These results parallel the results obtained in the pre-lab simulations, and correlate to the values of convergence shown in the bifurcation diagram provided to us.

Figure 21: Reaching Stability by Setting A=1.85V

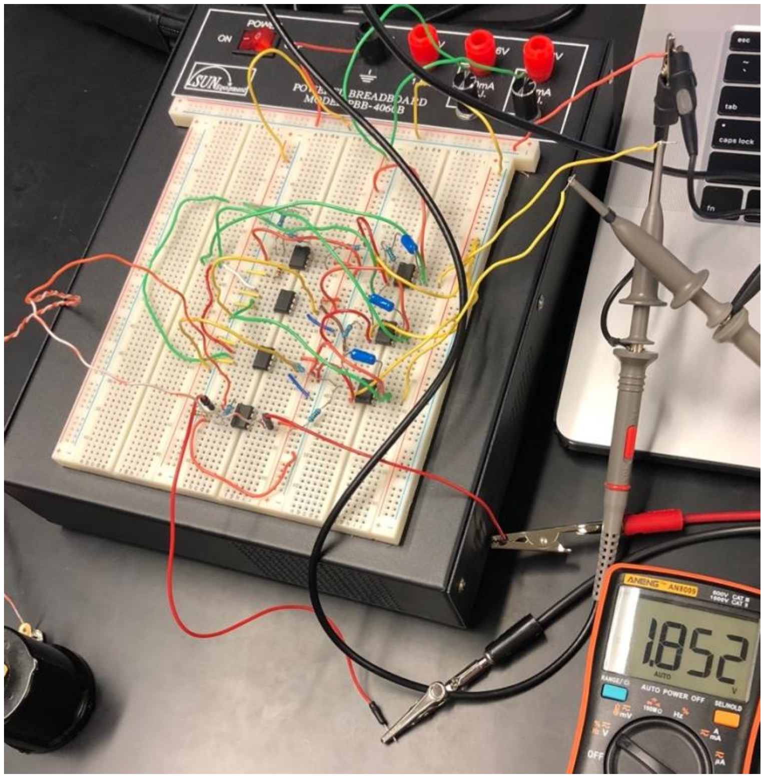


Figure 22: Phase Plane-Trajectory of x(t) vs. z(t), A = 1.85V

Step G: Resetting Initial Conditions of Circuit

In step G, we were asked to turn the system off, and turn it on again and observe the consequences this had on the circuit. By doing this, we are effectively randomizing the initial conditions of the circuit. The consequence of this was that, when turning the circuit back on and having the same value of constant A set as in step G, the phase-plane trajectory of the output signals no longer demonstrated convergence. The potentiometer had to be fine-tuned to a slightly different value for the circuit to demonstrate the same stable trajectory after each time the circuit was tuned off and turned on again.

Step H: Stability at A = 2.5

Figure 23 shows constant A set to approximately 2.5V. Experimentally, when setting A to 2.487V, the Lorenz circuit reached another point of convergence, shown in Figure 24. This once again matches the results we obtained in the simulations performed in the prelab activity.
When turning the circuit off and on again, as previously stated, the constant A had to be re-tuned to a slightly different value.

Figure 23: Setting A to approx. 2.5V

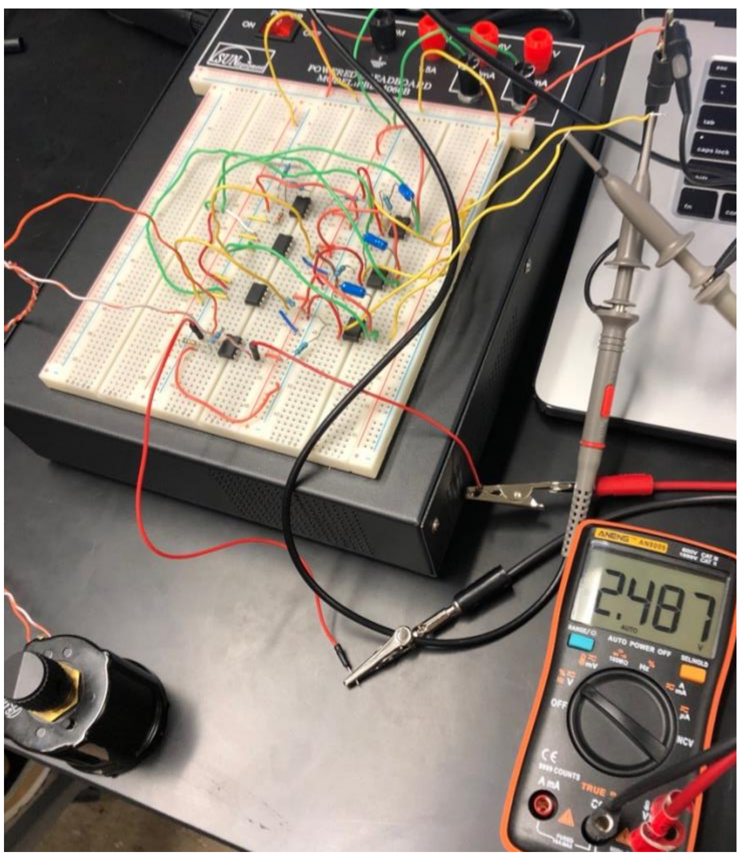
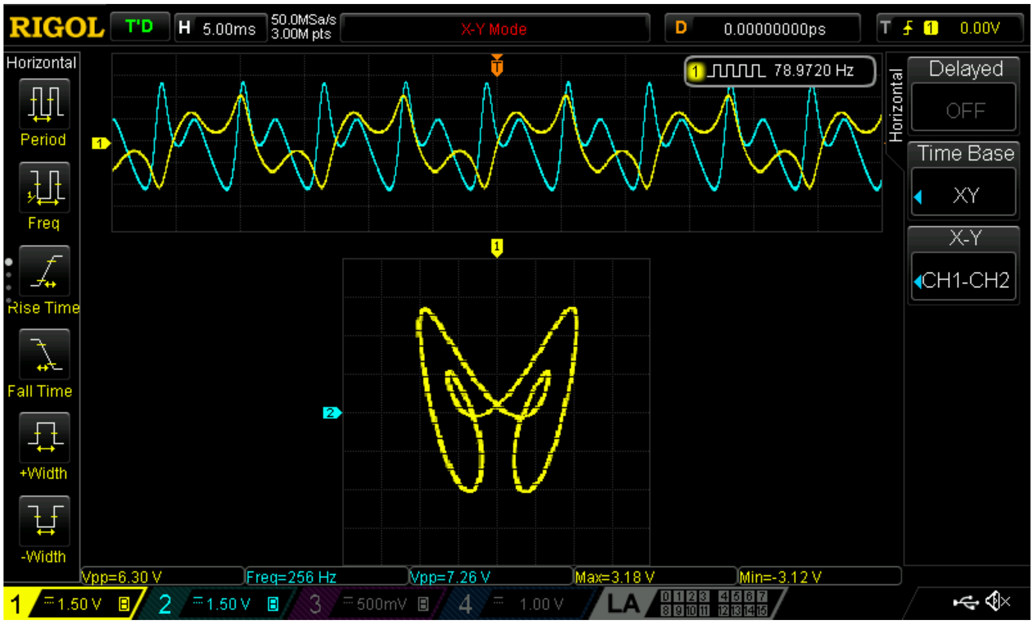


Figure 24: Phase Plane-Trajectory of x(t) vs. z(t), A = 2.5V



Step I: Stability at A = 2.5, y(t) vs x(t) and z(t) vs. y(t)

Figures 25 and 26 show the experimental results of the phase plane trajectories for output signals y(t) vs. x(t) and z(t) vs. y(t) of the Lorenz circuit, at constant A set to approximately 2.5V. The convergence of the system is evident once again here, as it should be. Nothing was changed in the circuit, only the signals across different points in the circuit were plotted using the oscilloscope.

Figure 25: y(t) vs. x(t), A = 2.5V

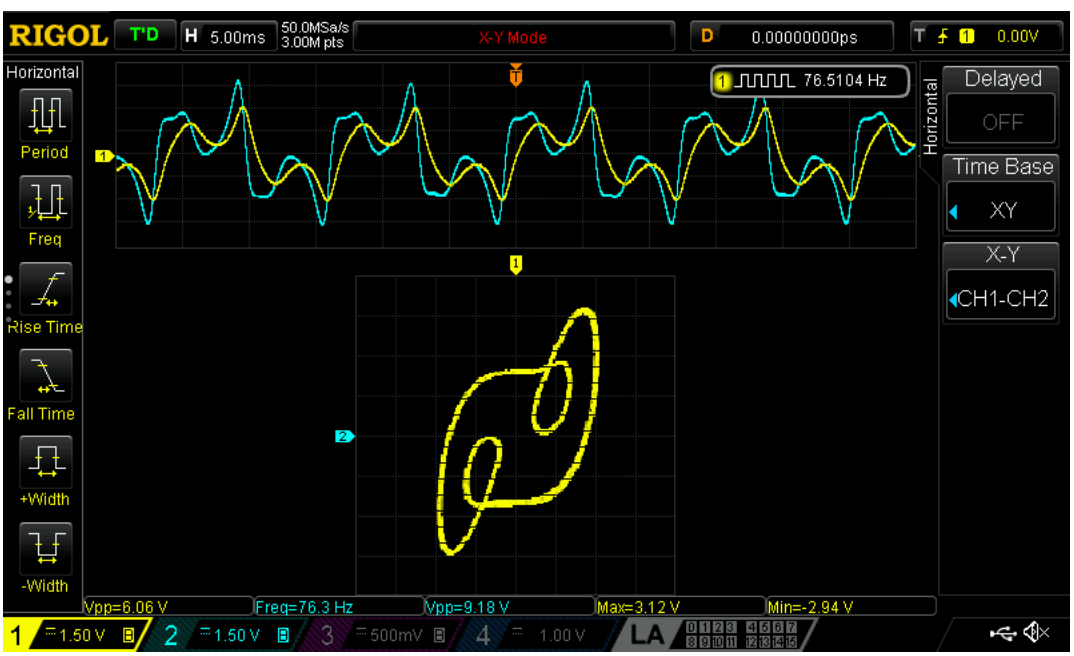


Figure 26:z(t) vs. y(t), A = 2.5V

Step I: Finding Stability for Various Values of A:

Figures 27 and 28 show the convergence of the Lorenz circuit for when constant A was set to approximately 5.2V. The experimental stability of the system at this constant value once again agrees with the values of convergence shown the in bifurcation table provided.

Figure 27: y(t) vs. x(t), A = 5.2V

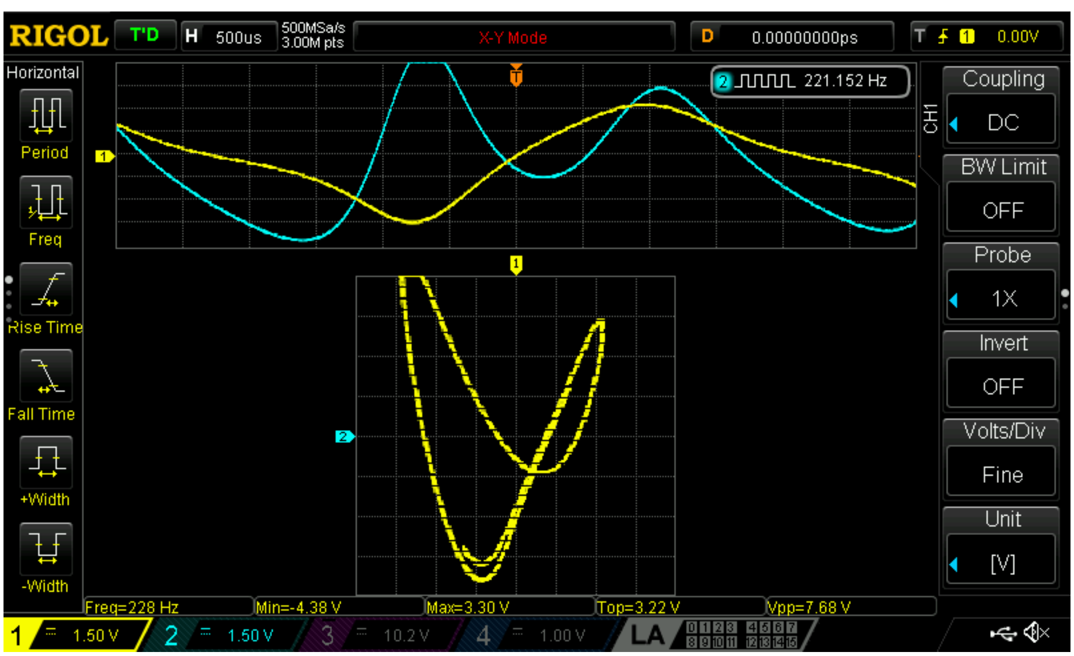
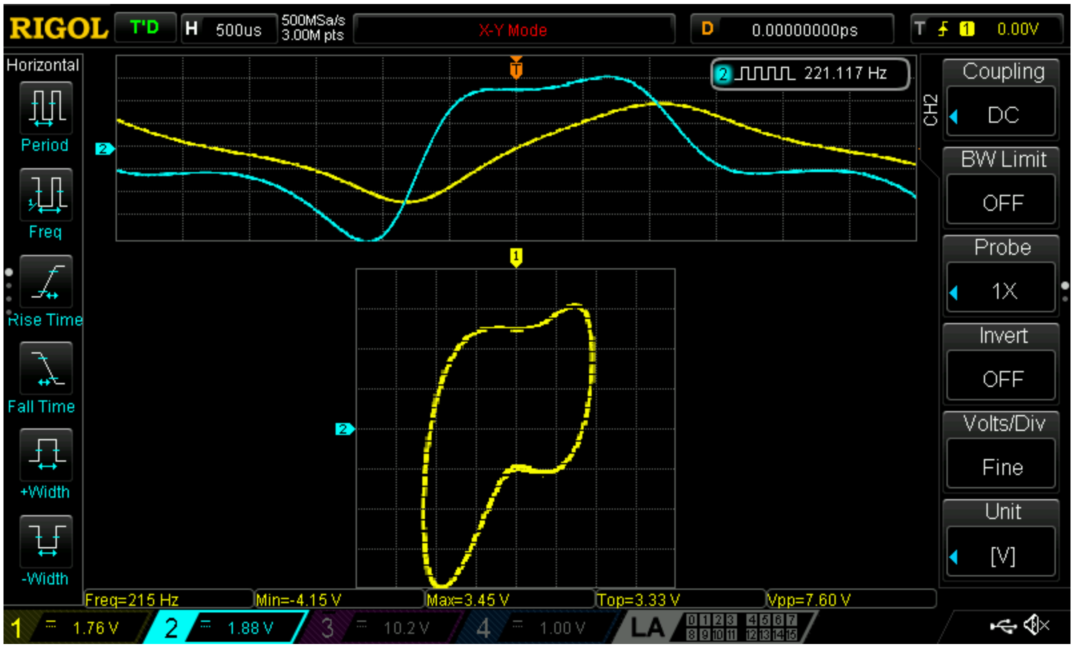


Figure 28: z(t) vs. y(t), A = 5.2V



Figures 29 and 30 show the phase plane trajectories of y(t) vs. x(t) and z(t) vs. y(t), respectively, when constant A was set to approximately 0.550V. Unfortunately, the stability of the trajectories is not evident in the images; the convergence of the output signals was very brief at this value of A, and even simply nudging the circuit board caused large disturbances in the output. I hypothesize that, at such small values of A, the impulse that A causes in the system was not large enough to overcome the noise of the experimental circuit.

Figure 29: y(t) vs. x(t), A = 0.550V

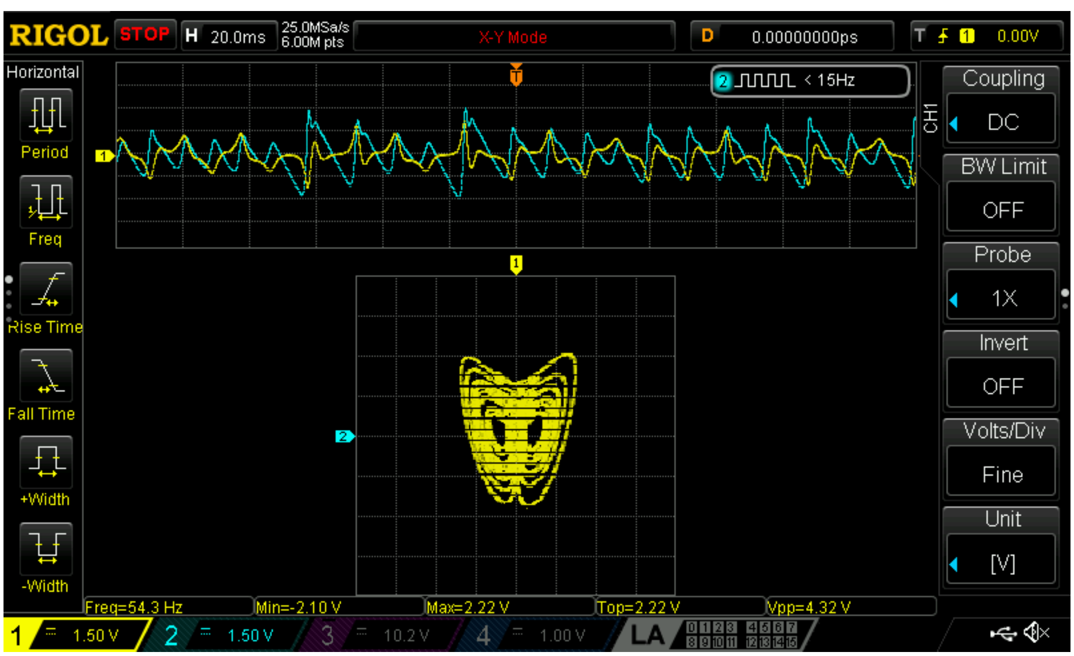
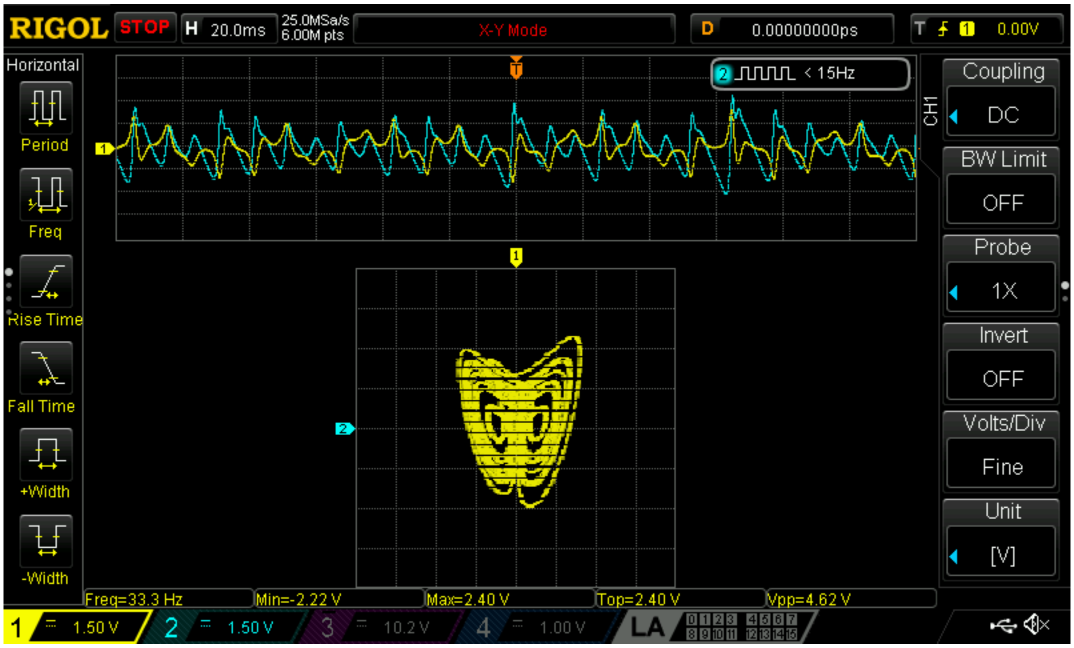


Figure 30: z(t) vs. y(t), A = 0.550V



IV: Conclusion

To conclude, ECE 4331 Experiment 5 provided us with an invaluable opportunity to delve into the intriguing world of chaotic systems, particularly through the lens of the Lorenz Diffusion-less Chaotic System. This experiment extended our understanding of analog circuits and showcased the profound impact that even minor component variations can have on a system's behavior.

Throughout the lab, we meticulously constructed and analyzed the analog circuit realization of the Lorenz system. By incorporating operational amplifiers, resistors, capacitors, and multipliers, we successfully replicated a system that exhibits chaotic behavior. One of the most insightful aspects of this lab was observing the effects of altering the integrator capacitors. By replacing them with larger values, we could directly witness the sensitivity of the Lorenz system to changes in circuit parameters.
Witnessing the transition from mathematical equations and simulations to an actual, functioning chaotic system was a testament to the power of analog electronics in modeling complex natural phenomena.

In summary, Experiment 5 was an rewarding journey into the realm of chaos theory and its practical implementation using analog electronics. It reinforced our comprehension of creating complex systems through synthesizing mathematical models via circuit design. The successful completion of this experiment and the insights gained from it are valuable additions to our repertoire as electrical engineering students.

Thank you!